

Arithmetic Central Extensions and Reciprocity Laws for Arithmetic Surface

K. Sugahara and L. Weng

Abstract

Three types of reciprocity laws for arithmetic surfaces are established. For these around a point or along a vertical curve, we first construct K_2 type central extensions, then introduce reciprocity symbols, and finally prove the law as an application of Parshin-Beilinson's theory of adelic complex. For reciprocity law along a horizontal curve, we first introduce a new type of arithmetic central extensions, then apply our arithmetic adelic cohomology theory and arithmetic intersection theory to prove the related reciprocity law.

All this can be interpreted within the framework of arithmetic central extensions. We add an appendix to deal with some basic structures of such extensions.

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1 Introduction

About 50 years ago, Tate [13] developed a theory of residues for curves using traces and adelic cohomologies. Tate's work was integrated with the K_2 central extensions by Arbarello-De Concini-Kac [1]. Reciprocity laws for algebraic surfaces were established by Parshin in [10]. Later, these reciprocity laws were reproved by Osipov [6], based on Kapranov's dimension theory [5]. Osipov constructs dimension two central extensions and hence establishes the reciprocity law for algebraic surfaces using Parshin's adelic theory [9]. More recently, a categorical proof of Parshin's reciprocity law was found by Osipov-Zhu [8]. In essence, Osipov's construction may be viewed as K_2 type theory of central extensions developed by Brylinski-Deligne [3].

However, to establish reciprocity laws for arithmetic surfaces, algebraic K_2 theory of central extensions is not sufficient: while it works for reciprocity laws around points and along vertical curves, it fails when we treat horizontal curves. To remedy this, instead, we first develop a new theory of arithmetic central extensions, based on the fact that for exact sequence of metrized \mathbb{R} -vector spaces of finite dimensional

$$\overline{V}_* : \quad 0 \rightarrow \overline{V}_1 \rightarrow \overline{V}_2 \rightarrow \overline{V}_3 \rightarrow 0,$$

there is a volume discrepancy $\gamma(\overline{V}_*)$. Accordingly, for an \mathbb{R} -vector space V , not necessary to be finite dimensional, a subspace A , and commensurable subspaces B, C , following [1], we have the group

$$\mathrm{GL}(V, A)(V, A) := \{g \in \mathrm{Aut}(V) : A \sim gA\},$$

the \mathbb{R} -line

$$(A|B) := \lambda(A/A \cap B)^* \otimes \lambda(B/A \cap B),$$

and the natural contraction isomorphism

$$\alpha := \alpha_{A,B,C} : (A|B) \otimes (B|C) \simeq (A|C).$$

Note that $A/A \cap B$ and $B/A \cap B$ are finite dimensional \mathbb{R} -spaces, we hence can introduce metrics on them and hence obtaining the metrized \mathbb{R} -line $\overline{(A|B)}$. In general, the contraction map $\alpha_{A,B,C}$ of (4.3) of [1] does not give an isometry. We denote the corresponding discrepancy by $\gamma(\overline{\alpha_{A,B,C}})$ and hence the isometry $\overline{\alpha}_{A,B,C} := \alpha_{A,B,C} \cdot \gamma(\overline{\alpha_{A,B,C}})$ so that we obtain a canonical isometry

$$\overline{\alpha} := \overline{\alpha}_{A,B,C} : \overline{(A|B)} \otimes \overline{(B|C)} \cong \overline{(A|C)}.$$

In parallel, for a pair of commensurable (metrized) subspaces A, B and A', B' , the natural isomorphism β introduced in '4.4) of [1] induced a canonical isometry

$$\overline{\beta}_{A,B;A',B'} : \overline{(A|B)} \otimes \overline{(A'|B')} \longrightarrow \overline{(A \cap A'|B \cap B')} \otimes \overline{(A + A'|B + B')}.$$

The up-shot of this consideration then leads to the following construction of arithmetic central extension $\widehat{\mathrm{GL}}(V, A)^{\mathrm{ar}}$ of $\mathrm{GL}(V, A)$ is given by

$$\widehat{\mathrm{GL}}(V, A)^{\mathrm{ar}} := \left\{ (g, a) : g \in \mathrm{GL}(V, A), a \in \overline{(A|gA)}, a \neq 0 \right\}$$

together with the multiplication

$$(g, a) \circ (g', a') := (gg', a \circ g(a'))$$

where

$$ab := a \circ b := \overline{\alpha}_{A,B,C}(a \otimes b) \in \overline{(A|C)} \quad \forall a \in \overline{(A|B)}, b \in \overline{(B|C)}.$$

Our first result is the following analogue of

Proposition A.

- (1) $(\widehat{\mathrm{GL}}(V, A)^{\mathrm{ar}}, \circ, (e, 1))$ forms a group;
- (2) There is a canonical central extension of groups

$$1 \rightarrow \mathbb{R}^* \xrightarrow{\iota} \widehat{\mathrm{GL}}(V, A)^{\mathrm{ar}} \xrightarrow{\pi} \mathrm{GL}(V, A) \rightarrow e.$$

Consequently, for elements $(g, a), (h, b) \in \widehat{\mathrm{GL}}(V, A)^{\mathrm{ar}}$, their commutator is given by

$$[(g, a), (h, b)] = ([g, h], a \circ g(b) \circ (ghg^{-1})(a^{-1}) \circ [g, h](b^{-1})).$$

In particular, from Theorem A(2), we have, if g and h commute,

$$\langle g, h \rangle := \langle g, h \rangle_A := a \circ g(b) \circ ghg^{-1}(a^{-1}) \circ [g, h](b^{-1}) \in \mathbb{R}^*.$$

We will show that $\langle g, h \rangle$ is well-defined. Moreover, we have the following

Proposition B. (Arithmetic Reciprocity Law) *Let A, B be two subspaces of V . For $g, h \in \mathrm{GL}(V, A) \cap \mathrm{GL}(V, B)$, we have*

$$\overline{\beta}(\langle g, h \rangle_A \otimes \langle g, h \rangle_B) = \langle g, h \rangle_{A \cap B} \otimes \langle g, h \rangle_{A+B}.$$

In particular, if g and g commutes, we have

$$\langle g, h \rangle_A \langle g, h \rangle_B = \langle g, h \rangle_{A \cap B} \langle g, h \rangle_{A+B}.$$

As a special case, if we assume that the metric system is rigid, that is to say, as to be defined in §?, we then recover the constructions of [1].

To go further, we consider the local field $\mathbb{R}((t))$ of Laurent series with \mathbb{R} coefficients. For each finite dimensional sub-quotient space of $\mathbb{R}((t))$, we may identify it with a standard form $V_{m,n} := \sum_{i=m}^n \mathbb{R}t^i$ for suitable integer

$m, n \in \mathbb{Z}$. We assign a metric on $V_{m,n}$ based on the standard Eulidean metric of \mathbb{R} with $(t^i, t^j) = \delta_{ij}, \forall m \leq i, j \leq n$. Then, for any $f(t), g(t) \in \mathbb{R}((t))$, if we write $f(t) = t^{\nu(f)} f_0(t)$, $g(t) = t^{\nu(g)} g_0(t)$, then $(f(t) \mathbb{R}[[t]] \mid g(t) \mathbb{R}[[t]])$ can be explicitly calculated. As a direct consequence, we have

$$\nu_{\mathbb{R}((t))}(f, g) := \log \langle f, g \rangle_{\mathbb{R}[[t]]} = \log \frac{|f_0(0)^{\nu(g)}|}{|g_0(0)^{\nu(f)}|}.$$

The pairing $\langle f, g \rangle_{\mathbb{R}[[t]]}$ is in fact the reciprocity symbol along horizontal curve at infinity. To introduce reciprocity symbol at finite places, we start with corresponding 2 dimensional local fields $L, k_L((u)), k_L\{\{u\}\}$ with k_L finite extensions of \mathbb{Q}_P . On L , there exists a discrete valuation of rank 2: $(\nu_1, \nu_2) : L^* \rightarrow \mathbb{Z} \oplus \mathbb{Z}$. With the help of this, then we can define a reciprocity symbol

$$\nu_L : K_2(L) \rightarrow \mathbb{Z}, \quad (f, g) \mapsto \begin{vmatrix} \nu_1(f) & \nu_1(g) \\ \nu_2(f) & \nu_2(g) \end{vmatrix}.$$

With all this, we are now ready to state our reciprocity laws. Let $\pi : X \rightarrow \text{Spec } \mathcal{O}_F$ be a regular arithmetic surface defined over a number field F with generic fiber X_F . Let C be an irreducible curve on X and x a closed point of X . As usual, see e.g., [10], we obtain an Artinian ring $K_{C,x}$ which is a finite direct sum of two dimensional local fields L_i . Accordingly, we define

$$\nu_{C,x} := \oplus_i [k_i : \mathcal{F}_q] \cdot \nu_{L_i}.$$

Our main theorem of this paper is the following

Main Theorem. (Reciprocity Law for Arithmetic Surfaces)

(1) For a fixed point $x \in X$,

$$\sum_{C: x \in C} \nu_{C,x}(f, g) = 0, \quad \forall f, g \in k(X)^*,$$

where C run over all irreducible curves on X which pass through x ;

(2) For a fixed vertical prime divisor V on X ,

$$\sum_{x: x \in V} \nu_{V,x}(f, g) = 0, \quad \forall f, g \in k(X)^*,$$

where x run over all closed points of X which lie on V ;

(3) For a fixed Horizontal prime divisor E_P on X corresponding an algebraic point P on X_F . For each infinite place σ of F , denote by $P_{\sigma,j}$ be corresponding closed points on X_{F_σ} . Then

$$\sum_{x: x \in E_P} \nu_{E_P,x}(f, g) \cdot \log q_x + \sum_{\sigma} \sum_j N_{\sigma} \cdot \nu_{P_{\sigma,j}}(f, g) = 0, \quad \forall f, g \in k(X)^*.$$

Our proof of this theorem is as follows: for the first two, similar as in [6], we first construct a K_2 -central extension of $k(X)^*$ with the help of Kapranov's dimension theory, then we interpret $\nu_{C,x}$ as a commutator of lifting of elements in some K_2 central extension of the group $K(X)^*$, and finally use the splitting of the central extension of adeles.

During this process, we discover that the fundamental reason for our reciprocity law along a vertical curve is the Riemann-Roch in dimension one and the intersection in dimension two. With the help of this, with a bit struggle, we finally can establish the reciprocity law for horizontal curves with a new construction of arithmetic central extension, arithmetic intersection in dimension two and a refined arithmetic Riemann-Roch theorem for arithmetic curves under the frame work of our arithmetic adelic theory built up in [11].

2 Reciprocity Laws in Dimension Two

2.1 Arithmetic Adelic Complex

Let P be an algebraic point of X_F . Denote by E_P the corresponding prime horizontal divisor of X , \mathcal{I}_{E_P} the ideal sheaf of $E_P \subset X$, and \overline{E}_P its arithmetic compactification. Let \mathcal{F} be a coherent sheaf on X . We introduce an arithmetic adelic complex $\mathcal{A}_{E_P,*}^{\text{ar}}(\mathcal{F})$ by

$$\mathcal{A}_{E_P,*}^{\text{ar}}(\mathcal{F}) := \varinjlim_n \varprojlim_{m: m \geq n} \mathbb{A}_{X,*}^{\text{ar}}(\mathcal{F} \otimes \mathcal{I}_{E_P}^n / \mathcal{I}_{E_P}^m),$$

where $\mathbb{A}_{X,*}^{\text{ar}}$ denotes the arithmetic adelic functor introduced in §1.2.3 of [11]. Since $\mathcal{A}_{E_P,*}^{\text{ar}}(\mathcal{F})$ is defined over an infinitesimal neighborhood of horizontal curve \overline{E}_P in \overline{X} , the complex consists of three terms. For example, when \mathcal{L} is an invertible sheaf, a direct calculation implies that $\mathcal{A}_{E_P,*}^{\text{ar}}(\mathcal{L})$ is given by

$$\widehat{k(X)}_{E_P} \times \left(\prod_{x \in E_P} (B_x \otimes_{\widehat{\mathcal{O}}_{X,x}} \mathcal{L}) \times (B_P \otimes_{\widehat{\mathcal{O}}_{X,P}} \mathcal{L} \widehat{\otimes}_{\mathbb{Q}} \mathbb{R}) \right) \longrightarrow \mathbb{A}_{\overline{E}_P}^{\text{ar}}$$

where $B_x := \mathcal{O}_{E_P,x}((u))$, $B_P = \mathcal{O}_{X_F}|_P((u))$ and $\mathbb{A}_{\overline{E}_P}^{\text{ar}} := \prod'_{x \in \overline{E}_P} \widehat{k(\overline{E}_P)}_x((u))$. That is, its elements are given by $\mathbf{a} = (a_x)_{x \in \overline{E}_P}$, where $a_x = \sum_i a_{i,x} u^i$ satisfying that, for any fixed i , $(a_{i,x})_{x \in \overline{E}_P}$ is an usual adele of the arithmetic curve \overline{E}_P . In particular, $\mathcal{A}_{E_P,*}^{\text{ar}}(\overline{\mathcal{O}}_X)$ is given by

$$\widehat{k(X)}_{E_P} \times \left(\prod_{x \in E_P} \mathcal{O}_{E_P,x}((u)) \times \mathcal{O}_{X_F}|_P((u)) \widehat{\otimes}_{\mathbb{Q}} \mathbb{R} \right) \longrightarrow \prod'_{x \in \overline{E}_P} \widehat{k(\overline{E}_P)}_x((u)).$$

Consequently, if we fix a non-zero rational section s of \mathcal{L} such that s does not vanish on E_P , we may write $\mathcal{L}|_{E_P}$ as $\mathcal{O}_{E_P}(\sum_x n_x [x])$ and write $\mathcal{L}_F :=$

$\mathcal{L}|_{X_F}$ as $\mathcal{L}_F = \mathcal{O}_{X_F}(\sum_{Q \in X_F} m_Q[Q])$. Here $\sum_x n_x[x] = \text{div}(s|_{E_P})$ and $\sum_{Q \in X_F} m_Q[Q] = \text{div}(s|_{X_F})$. Accordingly, we obtain a complex

$$\widehat{k(X)}_{E_P} \times \left(\prod_{x \in E_P} \mathfrak{m}_{E_P, x}^{-n_x}((u)) \times \prod_{Q \in X_F} \mathcal{O}_{X_F}(Q)^{\otimes m_Q}|_P((u)) \widehat{\otimes}_{\mathbb{Q}} \mathbb{R} \right) \longrightarrow \prod'_{x \in \overline{E_P}} \widehat{k(E_P)}_x((u)).$$

Similarly, for a different non-zero section s' of \mathcal{L} such that s does not vanish on E_P , we may write $\mathcal{L}|_{E_P} = \mathcal{O}_{E_P}(\sum_x n'_x[x])$ and $\mathcal{L}_F = \mathcal{O}_{X_F}(\sum_{Q \in X_F} m'_Q[Q])$. Hence, we have the corresponding complex

$$\widehat{k(X)}_{E_P} \times \left(\prod_{x \in E_P} \mathfrak{m}_{E_P, x}^{-n'_x}((u)) \times \prod_{Q \in X_F} \mathcal{O}_{X_F}(Q)^{\otimes m'_Q}|_P((u)) \widehat{\otimes}_{\mathbb{Q}} \mathbb{R} \right) \longrightarrow \prod'_{x \in \overline{E_P}} \widehat{k(E_P)}_x((u)).$$

2.2 Numerations in terms of Arakelov intersection

2.2.1 Case I

For a metrized line bundle $\overline{\mathcal{L}}$ on X , set

$$W_{\overline{\mathcal{L}}}^{\text{ar}} := \left(\prod_{x \in E_P} B_x \otimes_{\widehat{\mathcal{O}}_{X, x}} \mathcal{L} \right) \times \left(B_P \otimes_{\widehat{\mathcal{O}}_{X, x}} \overline{\mathcal{L}} \widehat{\otimes}_{\mathbb{Q}} \mathbb{R} \right).$$

To count it, for the non-zero rational section s of \mathcal{L} as above, we set

$$W_{\overline{\mathcal{L}}, s}^{\text{ar}} := \left(\prod_{x \in E_P} \mathfrak{m}_{E_P, x}^{-n_x}((u)) \right) \times \left(\prod_{Q \in X_F} \mathcal{O}_{X_F}(Q)^{\otimes m_Q}|_P((u)) \widehat{\otimes}_{\mathbb{Q}} \mathbb{R} \right) \times \mathbf{e} \left(\int_{X_\infty} -\log \|s\| d\mu \right).$$

Similarly, for the section s' as above, we get

$$W_{\overline{\mathcal{L}}, s'}^{\text{ar}} := \left(\prod_{x \in E_P} \mathfrak{m}_{E_P, x}^{-n'_x}((u)) \right) \times \left(\prod_{Q \in X_F} \mathcal{O}_{X_F}(Q)^{\otimes m'_Q}|_P((u)) \widehat{\otimes}_{\mathbb{Q}} \mathbb{R} \right) \times \mathbf{e} \left(\int_{X_\infty} -\log \|s'\| d\mu \right).$$

In particular, for a metrized line bundle $\overline{\mathcal{L}}_1$ satisfying $\mathcal{L}_1 \hookrightarrow \mathcal{L}$ with a non-zero rational section s_1 of \mathcal{L}_1 which does not vanish along E_P , we have

$$\begin{aligned} W_{\overline{\mathcal{L}}, s}^{\text{ar}} / W_{\overline{\mathcal{L}}_1, s_1}^{\text{ar}} &= \left(\prod_{x \in E_P} \mathfrak{m}_{E_P, x}^{-n_x + n_{1, x}}((u)) \right) \\ &\quad \times \left(\prod_{Q \in X_F} \mathcal{O}_{X_F}(Q)^{\otimes m_Q - m_{1, Q}}|_P((u)) \widehat{\otimes}_{\mathbb{Q}} \mathbb{R} \right) \\ &\quad \times \mathbf{e} \left(\int_{X_\infty} -\log \frac{\|s\|}{\|s_1\|_1} d\mu \right). \end{aligned}$$

To numerate them, we use Arakelov intersection [4]. Let $g(P, Q)$ be the Rrakelov-Green function and $G(P, Q) = e^{g(P, Q)}$. Then we introduce the following

Definition 1. (1) $\text{num}_0(\mathfrak{m}_{E_P, x}) := -\log q_x$;

(2) $\text{num}_0(\mathcal{O}_{X_F}(Q)|_P) := g(Q, P)$;

(3) $\text{num}_0(\mathcal{O}_{E_P, x}[[u]]/u^n \mathcal{O}_{E_P, x}[[u]]) := n$;

(4) $\text{num}_0(\mathcal{O}_{X_F, P}[[u]]/u^m \mathcal{O}_{X_F, P}[[u]]) := m$.

Proposition 1. Let $\overline{\mathcal{L}}_1$, resp. $\overline{\mathcal{L}}$, be a motorized line bundle on X satisfying $\mathcal{L} \hookrightarrow \mathcal{L}_1$. And let s_1 and s'_1 , resp. s and s' , be two non-zero rational sections of \mathcal{L}_1 , resp. of \mathcal{L} , which do not vanish along E_P . There exist canonical isometries of metrized \mathbb{R} -torsors

$$\begin{aligned} \text{Num}(W_{\overline{\mathcal{L}}, s}^{\text{ar}}) &\cong \text{Num}(W_{\overline{\mathcal{L}}, s'}^{\text{ar}}), \\ \text{Num}(W_{\overline{\mathcal{L}}_1, s_1}^{\text{ar}}/W_{\overline{\mathcal{L}}, s}^{\text{ar}}) &\cong \text{Num}(W_{\overline{\mathcal{L}}_1, s'_1}^{\text{ar}}/W_{\overline{\mathcal{L}}, s'}^{\text{ar}}). \end{aligned}$$

Proof. With the construction above, our proof for two isometries are similar. We here only give the details for the first one. Clearly, the numerations for the Laurent series parts of both sides are the same, we only need to deal with the coefficient part of both side. Hence, it suffices to show that

$$\begin{aligned} &\text{num}_0\left(\left(\prod_{x \in E_P} \mathfrak{m}_{E_P, x}^{-n_x}\right) \times \left(\prod_{Q \in X_F} \mathcal{O}_{X_F}(Q)^{\otimes m_Q}|_P \widehat{\otimes}_{\mathbb{Q}} \mathbb{R}\right) \times \mathbf{e}\left(\int_{X_\infty} -\log \|s\| d\mu\right)\right) \\ &= \text{num}_0\left(\left(\prod_{x \in E_P} \mathfrak{m}_{E_P, x}^{-n'_x}\right) \times \left(\prod_{Q \in X_F} \mathcal{O}_{X_F}(Q)^{\otimes m'_Q}|_P \widehat{\otimes}_{\mathbb{Q}} \mathbb{R}\right) \times \mathbf{e}\left(\int_{X_\infty} -\log \|s'\| d\mu\right)\right). \end{aligned}$$

By definition, the left hand side is equal to the logarithm of

$$\prod_{x \in E_P} q_x^{n_x} \prod_{Q \in X_F} G_\infty(P, \text{div}(s_F)) \cdot \mathbf{e}\left(\int_{X_\infty} -\log \|s\| d\mu\right),$$

which, by the Arakelov intersection theory [4], is simply equal to $\deg_{\text{ar}}(\overline{\mathcal{L}}|_{\overline{E_P}}) = c_{1, \text{ar}}(\overline{\mathcal{L}}) \cdot \overline{E_P}$. Similarly, the right hand side is equal to the logarithm of

$$\prod_{x \in E_P} q_x^{n'_x} \prod_{Q \in X_F} G_\infty(P, \text{div}(s'_F)) \cdot \mathbf{e}\left(\int_{X_\infty} -\log \|s'\| d\mu\right),$$

which is also equal to $\deg_{\text{ar}}(\overline{\mathcal{L}}|_{\overline{E_P}})$. \square

Definition 2. For metrized line bundles $\overline{\mathcal{L}}_i$ with non-zero rational sections s_i, s'_i ($i=1, 2$), which do not vanish along E_P , we define a metrized \mathbb{R} -torsor $[W_{\overline{\mathcal{L}}_1, s_1}^{\text{ar}}|W_{\overline{\mathcal{L}}_2, s_2}^{\text{ar}}]_2$ by

$$[W_{\overline{\mathcal{L}}_1, s_1}^{\text{ar}}|W_{\overline{\mathcal{L}}_2, s_2}^{\text{ar}}]_2 := \lim_{\substack{(\overline{\mathcal{L}}, s) \\ \mathcal{L} \hookrightarrow \mathcal{L}_1, \mathcal{L}_2}} \text{Hom}_{\mathbb{R}}\left(\text{Num}(W_{\overline{\mathcal{L}}_1, s_1}^{\text{ar}}/W_{\overline{\mathcal{L}}, s}^{\text{ar}}), \text{Num}(W_{\overline{\mathcal{L}}_2, s_2}^{\text{ar}}/W_{\overline{\mathcal{L}}, s}^{\text{ar}})\right).$$

Proposition 2. *With the same notation as above, we have a natural isometry of metrized \mathbb{R} -torsors*

$$[W_{\overline{\mathcal{L}}_1, s_1}^{\text{ar}} | W_{\overline{\mathcal{L}}_2, s_2}^{\text{ar}}]_2 = [W_{\overline{\mathcal{L}}_1, s'_1}^{\text{ar}} | W_{\overline{\mathcal{L}}_2, s'_2}^{\text{ar}}]_2.$$

In particular, the space $[W_{\overline{\mathcal{L}}_1}^{\text{ar}} | W_{\overline{\mathcal{L}}_2}^{\text{ar}}]_2$ is well-defined.

Proof. This is a direct consequence of the following calculation

$$\begin{aligned} & [W_{\overline{\mathcal{L}}_1, s_1}^{\text{ar}} | W_{\overline{\mathcal{L}}_2, s_2}^{\text{ar}}]_2 \\ &= \varprojlim_{\substack{(\overline{\mathcal{L}}, s) \\ \mathcal{L} \hookrightarrow \mathcal{L}_1, \mathcal{L}_2}} \text{Hom}_{\mathbb{R}} \left(\text{Num}(W_{\overline{\mathcal{L}}_1, s_1}^{\text{ar}} / W_{\overline{\mathcal{L}}, s}^{\text{ar}}), \text{Num}(W_{\overline{\mathcal{L}}_1, s_1}^{\text{ar}} / W_{\overline{\mathcal{L}}, s}^{\text{ar}}) \right) \\ &\cong \varprojlim_{\substack{(\overline{\mathcal{L}}', s') \\ \mathcal{L}' \hookrightarrow \mathcal{L}_1, \mathcal{L}_2}} \text{Hom}_{\mathbb{R}} \left(\text{Num}(W_{\overline{\mathcal{L}}_1, s'_1}^{\text{ar}} / W_{\overline{\mathcal{L}}', s'}^{\text{ar}}), \text{Num}(W_{\overline{\mathcal{L}}_2, s'_2}^{\text{ar}} / W_{\overline{\mathcal{L}}', s'}^{\text{ar}}) \right) \\ &\quad \quad \quad (\text{ by the second isometry of Prop. 1 above }) \\ &= [W_{\overline{\mathcal{L}}_1, s'_1}^{\text{ar}} | W_{\overline{\mathcal{L}}_2, s'_2}^{\text{ar}}]_2 \end{aligned}$$

□

2.2.2 Case II

Next, more generally, we deal with arbitrary non-zero rational sections s of \mathcal{L} , which may vanish along E_P .

Let f_0 be a non-zero rational function such that $s = s \cdot f_s^{-1}$ does not vanish along E_P . In particular, it makes sense for us to talk about $W_{\overline{\mathcal{L}}, s_0}^{\text{ar}}$.

To construct $W_{\overline{\mathcal{L}}, s}^{\text{ar}}$, write $s = s_0 \cdot u^{\nu_{E_P}(s)} \cdot f_0$ with f_0 a rational function and s_0 a section of the line bundle $\mathcal{L}_0 := \mathcal{L}(-\nu_{E_P}(s) \cdot E_P)$. There is a natural metric on \mathcal{L}_0 obtained as the tensor of $\overline{\mathcal{L}}$ and the $\nu_{E_P}(s)$ -th tensor power of $\mathcal{O}_X(\overline{E_P})$, namely, $\overline{\mathcal{L}}_1 := \overline{\mathcal{L}}(-\nu_{E_P}(s)\overline{E_P})$. Our idea of constructing $W_{\overline{\mathcal{L}}, s}^{\text{ar}}$ is to use the ‘existence’ of a natural decomposition

$$W_{\overline{\mathcal{L}}, s}^{\text{ar}} = W_{\overline{\mathcal{L}}_0, s_0}^{\text{ar}} \otimes W_{\mathcal{O}_X(\overline{E_P}), u}^{\otimes \nu_{E_P}(s)} \otimes W_{\overline{\mathcal{O}_X}, f_0}.$$

Since s_0 and f_0 do not vanish along E_P , the spaces $W_{\overline{\mathcal{L}}_0, s_0}^{\text{ar}}$ and $W_{\overline{\mathcal{O}_X}, f_0}$ make sense. Therefore, to construct $W_{\overline{\mathcal{L}}, s}^{\text{ar}}$ for a general non-zero section s , it suffices to define $W_{\mathcal{O}_X(\overline{E_P}), u}$ which we write simply as $W_{\overline{E_P}, u}$.

With the discussion of Case I above in mind, in particular, the role of Arakelov intersection, to construct $W_{\overline{E_P}, u}$, we first recall the Arakelov adjunction formula. Let \mathcal{K}_π be the canonical divisor of arithmetic surface $\pi : X \rightarrow Y := \text{Spac } \mathcal{O}_F$. Set $E_P = \text{Spec } A$. Then, see e.g., Cor. 5.5 at p.99 of [4], we have, globally,

$$\overline{\mathcal{K}}_\pi \cdot \overline{E_P} + \overline{E_P}^2 = \mathbf{d}_{E_P/Y} + \mathbf{d}_\lambda(\overline{E_P}).$$

Here

- (a) $\mathbf{d}_\lambda(\overline{E}_P) = \sum_{\sigma \in S_\infty} N_\sigma \mathbf{d}_{g,\sigma}(\overline{E}_P)$ with $\mathbf{d}_{g,\sigma}(\overline{E}_P) := \sum_{i \leq j} g(P_i, P_j)$ where $\{P_i\}$ are the collection of conjugating points on the Riemann surface X_σ corresponding to the algebraic point P of X_F ; and
- (b) $\mathbf{d}_{E_P/Y} = -\log(W_{E_P/Y} : \mathcal{O}_F)$, where $W_{E_P/Y}$ is the dualizing module of E_P over Y , defined by the fractional ideal of $F(P)$

$$W_{E_P/Y} := \{b \in F(P) : \text{Tr}(bA) \subset \mathcal{O}_F\}.$$

For later use, we write $W_{E_P/Y} := \prod_{x \in E_P} \mathfrak{m}_{E_P,x}^{b_x}$.

Moreover, by residue theorem, i.e., Theorem 4.1 of Chapter IV in [4], we know that the natural residue map induces a canonical isomorphism

$$\text{res} : \mathcal{K}_\pi(E_P)|_{E_P} \simeq \widetilde{W}_{E_P/Y},$$

where $\widetilde{W}_{E_P/Y}$ denotes the sheaf on affine curve E_P associated to the A -module $W_{E_P/Y}$.

Now we are ready to define $W_{\overline{E}_P,u}^{\text{ar}}$. As in the definition of $W_{\mathcal{L},s}$ in Case I above, $W_{\overline{E}_P,u}^{\text{ar}}$ admits a natural decomposition:

$$W_{\overline{E}_P,u}^{\text{ar}} = W_{E_P,u}^{\text{fin}} \times W_{\overline{E}_P,u}^\infty \cdot \mathbf{e}(c_{\overline{E}_P}).$$

It is too naive to set

$$W_{E_P,u}^{\text{fin}} = \prod_{x \in E_P} u \cdot \mathcal{O}_{E_P,x}((u)) \times \prod_{Q \in X_F} u^{\nu_P(f)} \mathcal{O}_{X_F}|_P((u)).$$

A bit thought in terms of Arakelov intersection, in particular, the adjunction formula and the associated residue theorem, leads to the use of the formula

$$W_{\overline{E}_P,u}^{\text{ar}} \otimes W_{\overline{\mathcal{K}}_\pi, \omega_0} = W_{\overline{K}_\pi(\overline{E}_P)}^{\text{ar}}.$$

Here ω_0 is a certain non-zero rational section of \mathcal{K}_π which does not vanish along E_P . Since, by the residue theorem recalled above, $\mathcal{K}_\pi(E_P)|_{E_P} = \widetilde{W}_{E_P/Y}$, accordingly, we define

$$\begin{aligned} W_{E_P,u}^{\text{fin}} &:= \prod_{x \in E_P} u \cdot \mathfrak{m}_{E_P,x}^{b_x - \nu_x(\omega_0|_{E_P})}((u)), \\ W_{\overline{E}_P,u}^\infty &:= \prod_{Q \in X_F} u \cdot \mathcal{O}_{X_F}(Q)^{\otimes -\nu_Q(\omega_0|_{X_F})}|_P((u)) \widehat{\otimes}_{\mathbb{Q}} \mathbb{R}, \\ c_{\overline{E}_P} &:= \int_{X_\infty} \log \|\omega_0\| d\mu - \mathbf{d}_\lambda(\overline{E}_P). \end{aligned}$$

Such defined, using the adjunction formula and the residue theorem again, we have

$$\begin{aligned}
& \text{num}_0 \left(\prod_{x \in E_P} \mathfrak{m}_{E_P, x}^{b_x - \nu_x(\omega_0|_{E_P})} \times \prod_{Q \in X_F} \cdot \mathcal{O}_{X_F}(Q)^{\otimes -\nu_Q(\omega_0|_{X_F})} \Big|_P \widehat{\otimes}_{\mathbb{Q}} \mathbb{R} \right. \\
& \quad \left. \cdot \mathbf{e} \left(\int_{X_\infty} \log \|\omega_0\| d\mu - \mathbf{d}_\lambda(\overline{E}_P) \right) \right) \\
& = -\mathbf{d}_{E_P/Y} + \overline{\mathcal{K}}_\pi \cdot \overline{E}_P - \mathbf{d}_\lambda(\overline{E}_P) = \overline{\mathcal{K}}_\pi \cdot \overline{E}_P - (\overline{\mathcal{K}}_\pi + \overline{E}_P) \cdot \overline{E}_P \\
& = \overline{E}_P \cdot \overline{E}_P.
\end{aligned}$$

All in all, we are ready to introduce the following

Definition 3. Let $\overline{\mathcal{L}}$ be a motorized line bundle on X with s a non-zero rational section. With respect to a decompose $s = s_0 \cdot u^{\nu_{E_P}(s)} \cdot f_0$, we define

$$W_{\overline{\mathcal{L}}, s}^{\text{ar}} := (W_{\overline{E}_P, u}^{\text{ar}})^{\otimes \nu_P(f_s)} \otimes W_{\overline{\mathcal{L}} \otimes \overline{E}_P^{\otimes -\nu_P(f_s)}, f_0 s_0}^{\text{ar}}.$$

Then the numeration of the coefficient part is equal to

$$\left(\nu_P(f_s) \overline{E}_P + c_{1, \text{ar}}(\overline{\mathcal{L}} \otimes \overline{E}_P^{\otimes -\nu_P(f_s)}) + c_{1, \text{ar}}(\mathcal{O}_X) \right) \cdot \overline{E}_P = c_{1, \text{ar}}(\overline{\mathcal{L}}) \cdot \overline{E}_P.$$

Hence we have proved the following

Theorem 3. For any non-zero rational sections s and s' of \mathcal{L} , we have

$$\text{Num}(W_{\overline{\mathcal{L}}, s}^{\text{ar}}) \cong \text{Num}(W_{\overline{\mathcal{L}}, s'}^{\text{ar}}).$$

In particular, we may write $\text{Num}(W_{\overline{\mathcal{L}}, s}^{\text{ar}})$ simply as $\text{Num}(W_{\overline{\mathcal{L}}}^{\text{ar}})$.

Consequently, we may introduce the following

Definition 4. For metrized line bundles $\overline{\mathcal{L}}_i$ with non-zero rational sections s_i, s'_i of \mathcal{L}_i ($i=1, 2$), we define a metrized \mathbb{R} -torsor $[W_{\overline{\mathcal{L}}_1, s_1}^{\text{ar}} | W_{\overline{\mathcal{L}}_2, s_2}^{\text{ar}}]_2$ by

$$[W_{\overline{\mathcal{L}}_1, s_1}^{\text{ar}} | W_{\overline{\mathcal{L}}_2, s_2}^{\text{ar}}]_2 := \lim_{\substack{\leftarrow \\ (\overline{\mathcal{L}}, s) \\ \mathcal{L} \hookrightarrow \mathcal{L}_1, \mathcal{L}_2}} \text{Hom}_{\mathbb{R}} \left(\text{Num}(W_{\overline{\mathcal{L}}_1, s_1}^{\text{ar}} / W_{\overline{\mathcal{L}}, s}^{\text{ar}}), \text{Num}(W_{\overline{\mathcal{L}}_2, s_2}^{\text{ar}} / W_{\overline{\mathcal{L}}, s}^{\text{ar}}) \right).$$

With the same proof as that for Proposition 2 in §2.2.1, we have proved the following

Theorem 4. With the same notation as above, we have a natural isometry of metrized \mathbb{R} -torsors

$$[W_{\overline{\mathcal{L}}_1, s_1}^{\text{ar}} | W_{\overline{\mathcal{L}}_2, s_2}^{\text{ar}}]_2 = [W_{\overline{\mathcal{L}}_1, s'_1}^{\text{ar}} | W_{\overline{\mathcal{L}}_2, s'_2}^{\text{ar}}]_2.$$

In particular, the space $[W_{\overline{\mathcal{L}}_1}^{\text{ar}} | W_{\overline{\mathcal{L}}_2}^{\text{ar}}]_2$ is well-defined.

2.3 Numerations in terms of arithmetic adelic cohomology

2.3.1 Case I

With the numeration in terms of intersection developed, next we introduce a numeration in terms of one dimensional cohomology theory. As we will see later reciprocity laws for arithmetic surfaces then can be proved using our refined arithmetic Riemann-Roch theorem [11] or better [12].

Proposition 5. *Let $\overline{\mathcal{L}}$ be a metrized invertible sheaf on X and s is a non-zero rational section of \mathcal{L} . Assume that s does not vanish along E_P , then the cohomology groups of the complex $\mathcal{A}_{E_P,*}^{\text{ar}}(\overline{\mathcal{L}}, s)$ is given by*

$$H^0(\mathcal{A}_{E_P,*}^{\text{ar}}(\overline{\mathcal{L}}, s)) = H_{\text{ar}}^0(\overline{E}_P, \overline{\mathcal{L}}|_{\overline{E}_P})((u)),$$

$$H^1(\mathcal{A}_{E_P,*}^{\text{ar}}(\overline{\mathcal{L}}, s)) = H_{\text{ar}}^1(\overline{E}_P, \overline{\mathcal{L}}|_{\overline{E}_P})((u)).$$

Proof. This is a direct consequence of arithmetic adelic cohomology theory developed in [11]. Indeed, for the invertible sheaf \mathcal{L} , by the definition of the complex $\mathcal{A}_{E_P,*}^{\text{ar}}(\overline{\mathcal{L}}, s)$, its cohomology is given by

$$\varinjlim_n \varprojlim_{m: m \geq n} H_{\text{ar}}^i(\overline{E}_P, \mathcal{L} \otimes \mathcal{I}_{E_P}^n / \mathcal{I}_{E_P}^m), \quad i = 0, 1.$$

This then already implies $H^0(\mathcal{A}_{E_P,*}^{\text{ar}}(\mathcal{L})) = H_{\text{ar}}^0(\overline{E}_P, \mathcal{L}|_{\overline{E}_P})((u))$. To prove $H^1(\mathcal{A}_{E_P,*}^{\text{ar}}(\mathcal{L})) = H_{\text{ar}}^1(\overline{E}_P, \mathcal{L}|_{\overline{E}_P})((u))$, we use the fact that, for horizontal curve \overline{E}_P , the maps

$$\varphi_{l,n;m} : H_{\text{ar}}^1(\overline{E}_P, \mathcal{L} \otimes \mathcal{I}_{E_P}^n / \mathcal{I}_{E_P}^m) \longrightarrow H_{\text{ar}}^1(\overline{E}_P, \mathcal{L} \otimes \mathcal{I}_{E_P}^l / \mathcal{I}_{E_P}^m)$$

are all injective, since Parshin-Beilinson's $H^0(E_P, \cdot)$ is exact. (In fact, since E_P is affine, Parshin-Beilinson's $H^1(E_P, \cdot)$ is always 0.) \square

Accordingly, we obtain two metrized \mathbb{R} -torsors

$$\text{Num}(H^0(\mathcal{A}_{E_P,*}^{\text{ar}}(\overline{\mathcal{L}}, s))) \quad \text{and} \quad \text{Num}(H^1(\mathcal{A}_{E_P,*}^{\text{ar}}(\overline{\mathcal{L}}, s))).$$

Definition 5. *For metrized line bundle $\overline{\mathcal{L}}$ on X and a non-zero rational section s of \mathcal{L} which does not vanish along E_P , we define a metrized \mathbb{R} -torsor $\text{Num}(\mathcal{A}_{E_P,*}^{\text{ar}}(\overline{\mathcal{L}}, s))$ by setting*

$$\begin{aligned} & \text{Num}(\mathcal{A}_{E_P,*}^{\text{ar}}(\overline{\mathcal{L}}, s)) \\ & := \text{Hom}_{\mathbb{R}}\left(\text{Num}(H^1(\mathcal{A}_{E_P,*}^{\text{ar}}(\overline{\mathcal{L}}, s))), \text{Num}(H^0(\mathcal{A}_{E_P,*}^{\text{ar}}(\overline{\mathcal{L}}, s)))\right). \end{aligned}$$

To go further, we need the following

Definition 6. We introduce a canonical numeration num_0 by

- (1) $\text{num}_0(H_{\text{ar}}^0(\overline{E}_P, \overline{\mathcal{L}}|_{\overline{E}_P})) := h_{\text{ar}}^0(\overline{E}_P, \overline{\mathcal{L}}|_{\overline{E}_P});$
- (2) $\text{num}_0(H_{\text{ar}}^1(\overline{E}_P, \overline{\mathcal{L}}|_{\overline{E}_P})) := h_{\text{ar}}^1(\overline{E}_P, \overline{\mathcal{L}}|_{\overline{E}_P});$
- (3) $\text{num}_0(A[[u]]/u^n A[[u]]) := n \cdot \text{num}_0(A)$ for any numerable locally compact space A

With all this, we are now ready to prove the following

Theorem 6. Let $\overline{\mathcal{L}}_i$ be metrized line bundle on X and s_i be a non-zero rational section of \mathcal{L}_i which does not vanish along E_P ($i=1,2$). There is a canonical isometry of metrized \mathbb{R} -torsors

$$[W_{\overline{\mathcal{L}}_1, s_1}^{\text{ar}} | W_{\overline{\mathcal{L}}_2, s_2}^{\text{ar}}]_2 \cong \text{Hom}_{\mathbb{R}}\left(\text{Num}(\mathcal{A}_{E_P, *}^{\text{ar}}(\overline{\mathcal{L}}_1, s_1)), \text{Num}(\mathcal{A}_{E_P, *}^{\text{ar}}(\overline{\mathcal{L}}_2, s_2))\right).$$

Proof. It suffices to identify special numerations in both sides canonically. Note that, for them, the Laurent series parts work in the same way, it suffices to treat their coefficient parts. From Prop ??, we know that num_0 for the left hand side gives

$$\deg_{\text{ar}}((\overline{\mathcal{L}}_1 - \overline{\mathcal{L}}_2)|_{\overline{E}_P}). \quad (*)$$

To complete the proof, it suffices to notice that num_0 for the coefficient part of $\text{Num}(\mathcal{A}_{E_P, *}^{\text{ar}}(\overline{\mathcal{L}}_1, s_1))$ equals

$$\begin{aligned} & h_{\text{ar}}^1(\overline{E}_P, \overline{\mathcal{L}}_1|_{\overline{E}_P}) - h_{\text{ar}}^0(\overline{E}_P, \mathcal{L}_1|_{\overline{E}_P}) \\ &= -\chi_{\text{ar}}(\overline{E}_P, \overline{\mathcal{L}}_1|_{\overline{E}_P}) = -\deg_{\text{ar}}((\overline{\mathcal{L}}_1)|_{\overline{E}_P}) - \frac{1}{2} \log |\Delta_{k(P)}|. \end{aligned}$$

Similarly, num_0 for the coefficient part of $\text{Num}(\mathcal{A}_{E_P, *}^{\text{ar}}(\overline{\mathcal{L}}_2, s_2))$ equals

$$\begin{aligned} & h_{\text{ar}}^1(\overline{E}_P, \overline{\mathcal{L}}_2|_{\overline{E}_P}) - h_{\text{ar}}^0(\overline{E}_P, \mathcal{L}_2|_{\overline{E}_P}) \\ &= -\chi_{\text{ar}}(\overline{E}_P, \overline{\mathcal{L}}_2|_{\overline{E}_P}) = -\deg_{\text{ar}}((\overline{\mathcal{L}}_2)|_{\overline{E}_P}) - \frac{1}{2} \log |\Delta_{k(P)}|. \end{aligned}$$

Therefore, the num_0 for the coefficient part of the right hand gives

$$\begin{aligned} & -\left(-\deg_{\text{ar}}((\overline{\mathcal{L}}_1)|_{\overline{E}_P}) - \frac{1}{2} \log |\Delta_{k(P)}|\right) + \left(-\deg_{\text{ar}}((\overline{\mathcal{L}}_2)|_{\overline{E}_P}) - \frac{1}{2} \log |\Delta_{k(P)}|\right) \\ &= \deg_{\text{ar}}((\overline{\mathcal{L}}_1 - \overline{\mathcal{L}}_2)|_{\overline{E}_P}). \end{aligned} \quad (**)$$

This coincides with the canonical numeration num_0 for the coefficient of the space $[W_{\overline{\mathcal{L}}_1, s_1}^{\text{ar}} | W_{\overline{\mathcal{L}}_2, s_2}^{\text{ar}}]_2$ obtained using Arakelov intersection given in (*). \square

2.3.2 Case II

With above discussion on special rational sections, next we treat arbitrary non-zero rational sections. Let then s be a non-zero rational section of \mathcal{L} and write $s = s_0 u^{\nu_P(f_s)} f_0$ with s_0 a rational section of $\mathcal{L}(-\nu_P(f_s)E_P)$, f_0 a rational function on X which does not vanish along E_P . Then, by definition,

$$W_{\overline{\mathcal{L}},s}^{\text{ar}} = (W_{\overline{E}_P,u}^{\text{ar}})^{\otimes \nu_P(f_s)} \otimes W_{\overline{\mathcal{L}} \otimes \overline{E}_P^{\otimes -\nu_P(f_s)},s_0}^{\text{ar}} \otimes W_{\overline{\mathcal{O}}_X,f_0}^{\text{ar}}.$$

And to calculate $H_{\text{ar}}^i(\mathcal{A}_{\overline{E}_P,*}^{\text{ar}}(\overline{\mathcal{L}},s)), i = 0, 1$, we use the complex

$$k(X)^* \times W_{\overline{\mathcal{L}},s}^{\text{ar}} \xrightarrow{\varphi} \mathbb{A}_{\overline{E}_P}^{\text{ar}}.$$

Here, if we write $\text{div}(s_0 f_0|_{E_P}) = \sum_x n_{0,x}[x]$, $\text{div}(s_0 f_0|_{X_F}) = \sum_Q m_{0,Q}[Q]$, by definition,

$$\begin{aligned} W_{\overline{\mathcal{L}},s}^{\text{ar}} &= \prod_{x \in E_P} u^{\nu_P(s)} \cdot \mathfrak{m}_{E_P,x}^{\nu_P(s)(b_x - \nu_x(\omega_0|_{E_P})) - n_{0,x}}((u)) \\ &\quad \times \prod_{Q \in X_F} u^{\nu_P(s)} \cdot \mathcal{O}_{X_F}(Q)^{-\nu_P(s)\nu_Q(\omega_0|_{X_F}) + m_{0,Q}}|_P((u)) \\ &\quad \cdot \mathbf{e}\left(\int_{X_\infty} (\nu_P(s) \log \|\omega_0\| - \log \|s_0 f_0\|) d\mu + \nu_P(s) \mathbf{d}_\lambda(\overline{E}_P)\right). \end{aligned}$$

Consequently, $H_{\text{ar}}^0(\mathcal{A}_{\overline{E}_P,*}^{\text{ar}}(\overline{\mathcal{L}},s)) = \text{Ker}(\varphi)$ which can be described as

$$\begin{aligned} H_{\text{ar}}^0(\mathcal{A}_{\overline{E}_P,*}^{\text{ar}}(\overline{\mathcal{L}},s)) &= u^{\nu_P(s)} \prod_{x \in E_P} \mathfrak{m}_{E_P,x}^{\nu_P(s)(b_x - \nu_x(\omega_0|_{E_P})) - n_{0,x}} \\ &\quad \times \prod_{Q \in X_F} G(Q,P)^{-\nu_P(s)\nu_Q(\omega_0|_{X_F}) + m_{0,Q}} \\ &\quad \cdot \mathbf{e}\left(\int_{X_\infty} (\nu_P(s) \log \|\omega_0\| - \log \|s_0 f_0\|) d\mu + \nu_P(s) \mathbf{d}_\lambda(\overline{E}_P)\right)((u)), \end{aligned}$$

and the corresponding quotient space $H_{\text{ar}}^1(\mathcal{A}_{\overline{E}_P,*}^{\text{ar}}(\overline{\mathcal{L}},s)) = \text{Coker}(\varphi)$, whose explicit description we leave to the reader. (See e.g. §1.2.4 of [11].)

Since $H_{\text{ar}}^i(\mathcal{A}_{\overline{E}_P,*}^{\text{ar}}(\overline{\mathcal{L}},s)), i = 0, 1$ are ind-pro topology spaces induced from Laurent series with coefficients numerable locally compact spaces, consequently, we obtain two metrized \mathbb{R} -torsors

$$\text{Num}(H^0(\mathcal{A}_{\overline{E}_P,*}^{\text{ar}}(\overline{\mathcal{L}},s))) \quad \text{and} \quad \text{Num}(H^1(\mathcal{A}_{\overline{E}_P,*}^{\text{ar}}(\overline{\mathcal{L}},s))).$$

As it stands, unlike in Def. 5, it is not easy to describe the numerations for the coefficient parts of them separately under num_0 . However, their difference can be treated well. For this we introduce the following

Definition 7. For metrized line bundle $\overline{\mathcal{L}}$ on X and a non-zero rational section s of \mathcal{L} , we define a metrized \mathbb{R} -torsor $\text{Num}(\mathcal{A}_{\overline{E}_P,*}^{\text{ar}}(\overline{\mathcal{L}}, s))$ by setting

$$\begin{aligned} & \text{Num}(\mathcal{A}_{\overline{E}_P,*}^{\text{ar}}(\overline{\mathcal{L}}, s)) \\ & := \text{Hom}_{\mathbb{R}}\left(\text{Num}(H^1(\mathcal{A}_{\overline{E}_P,*}^{\text{ar}}(\overline{\mathcal{L}}, s))), \text{Num}(H^0(\mathcal{A}_{\overline{E}_P,*}^{\text{ar}}(\overline{\mathcal{L}}, s)))\right). \end{aligned}$$

Lemma 7. With the same notation as above, the canonical numeration num_0 for the coefficient part of $\text{Num}(\mathcal{A}_{\overline{E}_P,*}^{\text{ar}}(\overline{\mathcal{L}}, s))$ is simply $-\chi_{\text{ar}}(\overline{E}_P, \overline{\mathcal{L}}|_{\overline{E}_P})$.

Proof. Indeed, by definition, the quantity we seek is equal to the negative of

$$\begin{aligned} & \chi_{\text{ar}}\left(\overline{E}_P, \prod_{x \in E_P} \mathfrak{m}_{E_P,x}^{\nu_P(s)(b_x - \nu_x(\omega_0|_{E_P})) - n_{0,x}} \right. \\ & \quad \times \prod_{Q \in X_F} G(Q, P)^{-\nu_P(s)\nu_Q(\omega_0|_{X_F}) + m_{0,Q}} \\ & \quad \left. \cdot \mathbf{e}\left(\int_{X_{\infty}} (\nu_P(s) \log \|\omega_0\| - \log \|s_0 f_0\|) d\mu + \nu_P(s) \mathbf{d}_{\lambda}(\overline{E}_P)\right)\right). \end{aligned}$$

That is,

$$\begin{aligned} & -\left(\nu_P(s) \mathbf{d}_{E_P/Y} - \nu_P(s) \cdot c_{1,\text{ar}}(\overline{\mathcal{K}}_{\pi}) \cdot \overline{E}_P + \nu_P(s) \mathbf{d}_{\lambda} \overline{E}_P \right. \\ & \quad \left. + (c_{1,\text{ar}}(\overline{\mathcal{L}}) - \nu_P \overline{E}_P) \cdot \overline{E}_P\right) + \frac{1}{2} \log |\Delta_{k(P)}| \\ & \quad \text{(by the residue theorem and definition)} \\ & = -\left(\nu_P(s)(c_{1,\text{ar}}(\overline{\mathcal{K}}_{\pi}) + \overline{E}_P) \cdot \overline{E}_P - \nu_P(s) \cdot c_{1,\text{ar}}(\overline{\mathcal{K}}_{\pi}) \cdot \overline{E}_P \right. \\ & \quad \left. + (c_{1,\text{ar}}(\overline{\mathcal{L}}) - \nu_P \overline{E}_P) \cdot \overline{E}_P\right) + \frac{1}{2} \log |\Delta_{k(P)}| \\ & \quad \text{(by the adjunction formula)} \\ & = -c_{1,\text{ar}}(\overline{\mathcal{L}}) \cdot \overline{E}_P + \frac{1}{2} \log |\Delta_{k(P)}| \\ & = -\chi_{\text{ar}}(\overline{E}_P, \overline{\mathcal{L}}|_{\overline{E}_P}) \\ & \quad \text{(by the Riemann-Roch formula).} \end{aligned}$$

□

With all this, we are now ready to prove the following

Theorem 8. Let $\overline{\mathcal{L}}_i$ be metrized line bundle on X and s_i be a non-zero rational section of \mathcal{L}_i ($i=1,2$). There is a canonical isometry of metrized \mathbb{R} -torsors

$$[W_{\overline{\mathcal{L}}_1,s_1}^{\text{ar}} | W_{\overline{\mathcal{L}}_2,s_2}^{\text{ar}}]_2 \cong \text{Hom}_{\mathbb{R}}\left(\text{Num}(\mathcal{A}_{\overline{E}_P,*}^{\text{ar}}(\overline{\mathcal{L}}_1, s_1)), \text{Num}(\mathcal{A}_{\overline{E}_P,*}^{\text{ar}}(\overline{\mathcal{L}}_2, s_2))\right).$$

Proof. It suffices to identify special numerations in both sides canonically. Note that, for both sides, the Laurent series parts work in the same way, it suffices to treat their coefficient parts. From our proof of Prop 3. and Thm. 4, we know that num_0 for the left hand side gives

$$\deg_{\text{ar}}((\overline{\mathcal{L}}_1 - \overline{\mathcal{L}}_2)|_{\overline{E}_P}). \quad (*)$$

To complete the proof, it suffices to notice that num_0 for the coefficient part of $\text{Num}(\mathcal{A}_{\overline{E}_P, *}^{\text{ar}}(\overline{\mathcal{L}}_1, s_1))$ gives $-\chi_{\text{ar}}(\overline{E}_P, \overline{\mathcal{L}}_1|_{\overline{E}_P})$. Indeed, similarly, num_0 for the coefficient part of $\text{Num}(\mathcal{A}_{\overline{E}_P, *}^{\text{ar}}(\overline{\mathcal{L}}_2, s_2))$ gives $-\chi_{\text{ar}}(\overline{E}_P, \overline{\mathcal{L}}_2|_{\overline{E}_P})$. Therefore, the num_0 for the coefficient part of the right hand side gives

$$\begin{aligned} & -\left(-\deg_{\text{ar}}((\overline{\mathcal{L}}_1)|_{\overline{E}_P}) + \frac{1}{2} \log |\Delta_{k(P)}|\right) + \left(-\deg_{\text{ar}}((\overline{\mathcal{L}}_2)|_{\overline{E}_P}) + \frac{1}{2} \log |\Delta_{k(P)}|\right) \\ & = \deg_{\text{ar}}((\overline{\mathcal{L}}_1 - \overline{\mathcal{L}}_2)|_{\overline{E}_P}). \end{aligned} \quad (**)$$

This coincides with the canonical numeration num_0 for the coefficient of the space $[W_{\overline{\mathcal{L}}_1, s_1}^{\text{ar}} | W_{\overline{\mathcal{L}}_2, s_2}^{\text{ar}}]_2$ obtained using Arakelov intersection given in (*). \square

2.4 Arithmetic Central Extension $k(X)_{W_{\overline{\mathcal{O}}_X}^{\text{ar}}}^*$

To define arithmetic central extension $k(X)_{W_{\overline{\mathcal{O}}_X}^{\text{ar}}}^*$, we first consider the action of $f \in k(X)^*$ on $W_{\overline{\mathcal{O}}_X}^{\text{ar}}$. Assume for the time being that f does not vanish along E_P . Set $\text{div}(f|_{E_P}) = \sum_x n_x [x]$, $\text{div}(f|_{X_F}) = \sum_Q m_Q [Q]$. In terms of pure algebraic structures involved,

$$W_{\overline{\mathcal{O}}_X}^{\text{ar}} = \prod_{x \in E_P} \mathcal{O}_{E_P, x}((u)) \times \prod_{Q \in X_F} \mathcal{O}_{X_F, Q}|_P((u)).$$

Thus, algebraically,

$$f \cdot W_{\overline{\mathcal{O}}_X}^{\text{ar}} = \prod_{x \in E_P} \mathfrak{m}_{E_P, x}^{-n_x}((u)) \times \prod_{Q \in X_F} \mathcal{O}_{X_F}(Q)^{m_Q}|_P((u)).$$

Accordingly, arithmetically, we define the action of f on $W_{\overline{\mathcal{O}}_X}^{\text{ar}}$ by

$$\begin{aligned} f W_{\overline{\mathcal{O}}_X}^{\text{ar}} &:= \prod_{x \in E_P} \mathfrak{m}_{E_P, x}^{-n_x}((u)) \times \prod_{Q \in X_F} \mathcal{O}_{X_F}(Q)^{m_Q}|_P((u)) \cdot \mathbf{e}\left(\int_{X_\infty} -\log \|f\| d\mu\right) \\ &= W_{\overline{\mathcal{O}}_X(\text{div}_{\text{ar}}(f)), f}^{\text{ar}}. \end{aligned}$$

Here $\text{div}_{\text{ar}}(f)$ denotes the Arakelov divisor associated to the rational function f , and we use $\overline{\mathcal{O}}_X(\text{div}_{\text{ar}}(f))$ to denote the metrized line bundle associated to the Arakelov divisor $\text{div}_{\text{ar}}(f)$.

Motivated by the above discussion, for an arbitrary non-zero rational function f , let the action of f on $W_{\mathcal{O}_X,1}^{\text{ar}}$ by

$$f \cdot W_{\mathcal{O}_X,1}^{\text{ar}} = W_{\mathcal{O}_X(\text{div}_{\text{ar}}(f)),f}^{\text{ar}}.$$

Then we have the following

Proposition 9. *For $f, g \in k(X)^*$, we have natural isometry*

$$[W_{\mathcal{O}_X}^{\text{ar}} : fW_{\mathcal{O}_X}^{\text{ar}}]_2 \otimes [fW_{\mathcal{O}_X}^{\text{ar}} : fgW_{\mathcal{O}_X}^{\text{ar}}]_2 \cong [W_{\mathcal{O}_X}^{\text{ar}} : fgW_{\mathcal{O}_X}^{\text{ar}}]_2.$$

Proof. Essentially, this is because arithmetic intersection works well. Indeed, by our definition, $fW_{\mathcal{O}_X}^{\text{ar}} = W_{\mathcal{O}_X(\text{div}_{\text{ar}}(f))}^{\text{ar}}$. Hence, it suffices to prove that there is an isometry

$$[W_{\mathcal{O}_X}^{\text{ar}} : W_{\mathcal{O}_X(\text{div}_{\text{ar}}(f))}^{\text{ar}}]_2 \otimes [W_{\mathcal{O}_X(\text{div}_{\text{ar}}(f))}^{\text{ar}} : W_{\mathcal{O}_X(\text{div}_{\text{ar}}(fg))}^{\text{ar}}]_2 \cong [W_{\mathcal{O}_X}^{\text{ar}} : W_{\mathcal{O}_X(\text{div}_{\text{ar}}(fg))}^{\text{ar}}]_2.$$

But, similarly, as in the proof of Prop. 2, this is clear by the definition of num_0 in terms of intersections, due to the following trivial cancelation

$$\text{div}_{\text{ar}}(f) + \left(-(\text{div}_{\text{ar}}(f)) + (\text{div}_{\text{ar}}(fg)) \right) = (\text{div}_{\text{ar}}(fg)).$$

That is,

$$\begin{aligned} & [W_{\mathcal{O}_X}^{\text{ar}} : W_{\mathcal{O}_X(\text{div}_{\text{ar}}(f))}^{\text{ar}}]_2 \otimes [W_{\mathcal{O}_X(\text{div}_{\text{ar}}(f))}^{\text{ar}} : W_{\mathcal{O}_X(\text{div}_{\text{ar}}(fg))}^{\text{ar}}]_2 \\ &= \varprojlim_{\substack{(\overline{\mathcal{L}},s), \mathcal{L} \hookrightarrow \mathcal{O}_X \\ \mathcal{L} \hookrightarrow \mathcal{O}_X(\text{div}_{\text{ar}}(f))}} \text{Hom}_{\mathbb{R}} \left(\text{Num}(W_{\mathcal{O}_X,1}^{\text{ar}}/W_{\overline{\mathcal{L}},s}^{\text{ar}}), \text{Num}(W_{\mathcal{O}_X(\text{div}_{\text{ar}}(f)),f}^{\text{ar}}/W_{\overline{\mathcal{L}},s}^{\text{ar}}) \right) \\ &\otimes \varprojlim_{\substack{(\overline{\mathcal{L}},s) \mathcal{L} \hookrightarrow \mathcal{O}_X(\text{div}_{\text{ar}}(f)) \\ \mathcal{L} \hookrightarrow \mathcal{O}_X(\text{div}_{\text{ar}}(fg))}} \text{Hom}_{\mathbb{R}} \left(\text{Num}(W_{\mathcal{O}_X(\text{div}_{\text{ar}}(f)),f}^{\text{ar}}/W_{\overline{\mathcal{L}},s}^{\text{ar}}), \text{Num}(W_{\mathcal{O}_X(\text{div}_{\text{ar}}(fg)),fg}^{\text{ar}}/W_{\overline{\mathcal{L}},s}^{\text{ar}}) \right) \\ &\cong \varprojlim_{\substack{(\overline{\mathcal{L}},s) \mathcal{L} \hookrightarrow \mathcal{O}_X, \mathcal{O}_X(\text{div}_{\text{ar}}(f)) \\ \mathcal{L} \hookrightarrow \mathcal{O}_X(\text{div}_{\text{ar}}(fg))}} \left(\text{Hom}_{\mathbb{R}} \left(\text{Num}(W_{\mathcal{O}_X,1}^{\text{ar}}/W_{\overline{\mathcal{L}},s}^{\text{ar}}), \text{Num}(W_{\mathcal{O}_X(\text{div}_{\text{ar}}(f)),f}^{\text{ar}}/W_{\overline{\mathcal{L}},s}^{\text{ar}}) \right) \right. \\ &\quad \left. \otimes \text{Hom}_{\mathbb{R}} \left(\text{Num}(W_{\mathcal{O}_X(\text{div}_{\text{ar}}(f)),f}^{\text{ar}}/W_{\overline{\mathcal{L}},s}^{\text{ar}}), \text{Num}(W_{\mathcal{O}_X(\text{div}_{\text{ar}}(fg)),fg}^{\text{ar}}/W_{\overline{\mathcal{L}},s}^{\text{ar}}) \right) \right) \\ &\cong \varprojlim_{\substack{(\overline{\mathcal{L}},s) \\ \mathcal{L} \hookrightarrow \mathcal{O}_X, \mathcal{O}_X(\text{div}_{\text{ar}}(fg))}} \text{Hom}_{\mathbb{R}} \left(\text{Num}(W_{\mathcal{O}_X,1}^{\text{ar}}/W_{\overline{\mathcal{L}},s}^{\text{ar}}), \text{Num}(W_{\mathcal{O}_X(\text{div}_{\text{ar}}(fg)),fg}^{\text{ar}}/W_{\overline{\mathcal{L}},s}^{\text{ar}}) \right) \\ &= [W_{\mathcal{O}_X}^{\text{ar}} : W_{\mathcal{O}_X(\text{div}_{\text{ar}}(fg))}^{\text{ar}}]_2. \end{aligned}$$

□

Consequently, we may make the following

Definition 8. *We define an arithmetic central extension group $k(X)_{W_{\mathcal{O}_X}^{\text{ar}}}^*$ by the following data:*

- (a) *As a set, its elements are given by pairs (f, α) with $f \in k(X)^*$, $\alpha \in [W_{\mathcal{O}_X}^{\text{ar}} : fW_{\mathcal{O}_X}^{\text{ar}}]_2$;*

(b) For the group law, its multiplication is defined by

$$(f, \alpha) \circ (g, \beta) := (fg, \alpha \circ f(\beta)),$$

where

$$\alpha \circ f(\beta) := \alpha \otimes f(\beta).$$

Indeed, since $\alpha \in [W_{\overline{\mathcal{O}}_X}^{\text{ar}} : fW_{\overline{\mathcal{O}}_X}^{\text{ar}}]_2$, $\beta \in [W_{\overline{\mathcal{O}}_X}^{\text{ar}} : gW_{\overline{\mathcal{O}}_X}^{\text{ar}}]_2$, we have

$$f(\beta) \in [fW_{\overline{\mathcal{O}}_X}^{\text{ar}} : fgW_{\overline{\mathcal{O}}_X}^{\text{ar}}]_2, \quad \text{and} \quad \alpha \circ f(\beta) \in [W_{\overline{\mathcal{O}}_X}^{\text{ar}} : fgW_{\overline{\mathcal{O}}_X}^{\text{ar}}]_2.$$

So (b) is well-defined.

2.5 Splitness

Concerning the group $k(X)_{W_{\overline{\mathcal{O}}_X}^{\text{ar}}}^*$, we have the following

Proposition 10. *We have*

- (1) *The group $k(X)_{W_{\overline{\mathcal{O}}_X}^{\text{ar}}}^*$ is a central extension of $k(X)^*$ by \mathbb{R} . In particular, we have an exact sequence*

$$0 \rightarrow \mathbb{R} \longrightarrow k(X)_{W_{\overline{\mathcal{O}}_X}^{\text{ar}}}^* \longrightarrow k(X)^* \rightarrow 1.$$

- (2) *The short exact sequence (1) splits.*

Proof. (1) follows directly from the definition. As for (2), we introduce one more central extension $k(X)_{W_{\overline{\mathcal{O}}_X}^{\text{ar}}}^{*'}$ of the group $k(X)^*$ by \mathbb{R} as follows:

- (a) As a set, its elements are given by pairs (f, α') , where $f \in k(X)^*$ and $\alpha' \in \text{Hom}_{\mathbb{R}}(\text{Num}(\mathcal{A}_{E_P, *}^{\text{ar}}(\overline{\mathcal{O}}_X), \text{Num}(\mathcal{A}_{E_P, *}^{\text{ar}}(f\overline{\mathcal{O}}_X)))$;

- (b) For the group law, its multiplication is defined by

$$(f, \alpha') \circ (g, \beta') := (fg, \alpha' \circ f(\beta')).$$

Now it is sufficient to use Theorem 8. Indeed, by Theorem 8, we obtain a natural isomorphism $k(X)_{W_{\overline{\mathcal{O}}_X}^{\text{ar}}}^* \simeq k(X)_{W_{\overline{\mathcal{O}}_X}^{\text{ar}}}^{*'}$. On the other hand, the central extension $k(X)_{W_{\overline{\mathcal{O}}_X}^{\text{ar}}}^{*'}$ splits, since it admits the following natural section: an element $h \in k(X)^*$ takes $\text{Num}(\mathcal{A}_{E_P, *}^{\text{ar}}(\overline{\mathcal{O}}_X))$ to $\text{Num}(\mathcal{A}_{E_P, *}^{\text{ar}}(\overline{\mathcal{O}}_X(\text{div}_{\text{ar}}(h))))$. That is, $h \in \text{Hom}_{\mathbb{R}}(\text{Num}(\mathcal{A}_{E_P, *}^{\text{ar}}(\overline{\mathcal{O}}_X)), \text{Num}(\mathcal{A}_{E_P, *}^{\text{ar}}(\overline{\mathcal{O}}_X(\text{div}_{\text{ar}}(h)))))$. This implies that the central extension $k(X)_{W_{\overline{\mathcal{O}}_X}^{\text{ar}}}^{*'}$ splits. \square

2.6 Reciprocity Symbols from Points and Curves

Let $\pi : X \longrightarrow \text{Spec } \mathcal{O}_F$ be a regular arithmetic surface defined over number field F . Let C be a complete irreducible vertical curve or an irreducible horizontal curve, and x a closed point of C . Set $H := \text{Frac}(\widehat{\mathcal{O}}_{X,x})^*$. Then with respect to the curve C at x , resp. the closed point x along C , there exists a natural central extension

$$0 \rightarrow \mathbb{R} \longrightarrow \widehat{H}_{B_x} \longrightarrow H \rightarrow 1 \quad (A)$$

resp.

$$0 \rightarrow \mathbb{R} \longrightarrow \widehat{H}_{\mathcal{O}_{K_{C,x}}} \longrightarrow H \rightarrow 1 \quad (B)$$

defined as follows:

- (a) As a set, its elements are given by pairs (f, α) , resp. by pairs (g, β) , with $f \in H$ and $\alpha \in [B_x | f B_x]_2$, resp. $g \in H$ and $\beta \in [\mathcal{O}_{K_{C,x}} | g \mathcal{O}_{K_{C,x}}]_2$;
- (b) For the group law, its multiplication is defined by

$$(f_1, \alpha_1) \circ (f_2, \alpha_2) = (f_1 f_2, \alpha_1 f_1(\alpha_2)), \text{ resp. } (g_1, \beta_1) \circ (g_2, \beta_2) = (g_1 g_2, \beta_1 g_1(\beta_2)).$$

Definition 9. For $f_1, f_2 \in H$, define a reciprocity symbol $[f_1, f_2]_{x,C}$, resp. $[f_1, f_2]_{C,x}$, by

$$[f_1, f_2]_{x,C} := [f'_1, f'_2]_A, \quad \text{resp.} \quad [f_1, f_2]_{C,x} := [f''_1, f''_2]_B,$$

where f'_1, f'_2 , resp. f''_1, f''_2 , are the lifts of elements f_1, f_2 of H via extension (A) to \widehat{H}_{B_x} , resp. via extension (B) to $\widehat{H}_{\mathcal{O}_{K_{C,x}}}$, and $[f'_1, f'_2]_A$, resp. $[f''_1, f''_2]_B$, denotes the commutative of elements f'_1, f'_2 , resp. f''_1, f''_2 , in \widehat{H}_{B_x} , resp. in $\widehat{H}_{\mathcal{O}_{K_{C,x}}}$.

We have the following

Proposition 11. For $f_1, f_2 \in H$,

$$[f_1, f_2]_{x,C} \cdot [f_1, f_2]_{C,x} = 1.$$

Proof. For vertical curves, this proposition is simply Prop 13 of [6]. For horizontal curves, the same proof works. Indeed, similarly, for any two invertible sheaves $\mathcal{L}, \mathcal{L}'$ satisfying $\mathcal{L} \subset \mathcal{L}'$, there exists a natural isomorphism

$$\begin{aligned} [B_x \otimes_{\widehat{\mathcal{O}}_{X,x}} \mathcal{L} | B_x \otimes_{\widehat{\mathcal{O}}_{X,x}} \mathcal{L}']_2 \otimes_{\mathbb{R}} [\mathcal{O}_{K_{C,x}} \otimes_{\widehat{\mathcal{O}}_{X,x}} \mathcal{L} | \mathcal{O}_{K_{C,x}} \otimes_{\widehat{\mathcal{O}}_{X,x}} \mathcal{L}']_2 \\ \longrightarrow \text{Hom}_{\mathbb{R}}(\text{Num}(\mathcal{A}_{C,x}(\mathcal{L})), \text{Num}(\mathcal{A}_{C,x}(\mathcal{L}'))). \end{aligned}$$

Consequently, if we introduce a third central extension $\widehat{H}_{B_x, \mathcal{O}_{K_{C,x}}}$ of H by:

- (a) As a set, its elements are given by pairs (f, γ) with $f \in H$ and $\gamma \in [B_x | f B_x]_2 \otimes_{\mathbb{R}} [\mathcal{O}_{K_{C,x}} | g \mathcal{O}_{K_{C,x}}]_2$;
- (b) For the group law, its multiplication is defined by

$$(f_1, \gamma_1) \circ (f_2, \gamma_2) = (f_1 f_2, \gamma_1 f_1(\gamma_2)).$$

Then $\widehat{H}_{B_x, \mathcal{O}_{K_{C,x}}}$ splits. □

2.7 Proof of Reciprocity Laws

We may use same proofs as in Theorem 2, resp. Theorem 3 of [6] to prove our reciprocity law around a point, resp. along an irreducible complete curve, for arithmetic surfaces, [6] works on algebraic surfaces over finite fields. This is because in this two cases above, only finite part of arithmetic surface is involved.

Thus, it suffices for us to prove reciprocity laws along horizontal curves for arithmetic surfaces.

For this purpose, we set

$$M_{E_P}^{\text{ar}} := \prod_{x \in E_P} K_{E_P, x} \times \prod_{Q \in X_F} K_{X_F, Q} |_P \widehat{\otimes}_{\mathbb{Q}} \mathbb{R}.$$

Then for $f, g \in k(X)^*$, set S_{E_P} to be the collection of places x of F such that x is in the union of the support of $\text{div}(f|_{E_P})$ and the support of $\text{div}(g|_{E_P})$, and let S_{X_F} to be the collection of points Q of X_F such that Q is in the union of the support of $\text{div}(f|_{X_F})$ and the support of $\text{div}(g|_{X_F})$. Let

$$M_{f,g}^{\text{ar}} := \prod_{x \in S_{E_P}} K_{E_P, x} \times \prod_{Q \in S_{X_F}} K_{X_F, Q} |_P \widehat{\otimes}_{\mathbb{Q}} \mathbb{R}$$

be a combination of factors of $M_{E_P}^{\text{ar}}$, and define $M_{f,g}^-$ to be its cofactor in $M_{E_P}^{\text{ar}}$ so that we have

$$M_{E_P}^{\text{ar}} = M_{f,g}^{\text{ar}} \times M_{f,g}^{-f,g}.$$

To go further, we use the central extension $k(X)^*_{W_{\widehat{\mathcal{O}}_X}^{\text{ar}}}$ of $k(X)^*$ with respect to $W_{\widehat{\mathcal{O}}_X}^{\text{ar}}$. By its splitness proved in Prop. 10 of §2.5, we have the associated reciprocity symbol $[\ast, \ast]_{W_{\widehat{\mathcal{O}}_X}^{\text{ar}}}$ vanishes. Similarly, we can also introduce the central extension of $k(X)^*$ with respect to $M_{f,g}^{\text{ar}}$ and $M_{f,g}^{-f,g}$ to get first the groups $k(X)^*_{M_{f,g}^{\text{ar}}}$ and $k(X)^*_{M_{f,g}^{-f,g}}$ and hence the associated reciprocity symbols $[\ast, \ast]_{M_{f,g}^{\text{ar}}}$ and $[\ast, \ast]_{M_{f,g}^{-f,g}}$. Based on the techniques developed in §2.2, it is rather direct to show that

$$[\ast, \ast]_{W_{\widehat{\mathcal{O}}_X}^{\text{ar}}} = [\ast, \ast]_{M_{f,g}^{\text{ar}}} + [\ast, \ast]_{M_{f,g}^{-f,g}}. \quad (*_3)$$

Moreover, since f, g keep $M_{\text{ar}}^{f,g}$ unchanged, we have

$$[f, g]_{M_{\text{ar}}^{f,g}} = 0 \quad \forall f, g \in k(X)^*.$$

Therefore, we have

$$0 = [f, g]_{W_{\mathcal{O}_X}^{\text{ar}}} = [f, g]_{M_{f,g}^{\text{ar}}} + [f, g]_{M_{\text{ar}}^{f,g}} = [f, g]_{M_{f,g}^{\text{ar}}}.$$

Indeed, for any factor subspace M of $W_{\mathcal{O}_X}^{\text{ar}}$, we can construct its associated central extension $k(X)_M^*$ and hence obtaining the reciprocity pairing $[\ast, \ast]_M$ such that $[\ast, \ast]$ is additive, and, if $f, g \in k(X)^*$ keep M stable, then, with the proof of Prop 4(4) of [6], we have $[f, g]_M = 0$. Consequently, by (\ast_3) ,

$$0 = \sum_{x \in E_P} [f, g]_{B_x} + \sum_{Q \in X_F} [f, g]_{\mathcal{O}_{X_F}(Q)}.$$

This is the abstract reciprocity law.

By Prop. 6 of §3.6, we have $[f, g]_{B_x} = [f, g]_{C,x} = \nu_{C,x}(f, g) \log q_x$. Hence, to complete the proof, it suffices to show that

$$[f, g]_{\mathcal{O}_{X_F}(Q)} = \log \frac{|f_0(P)^{\nu_P(g)}|}{|g_0(P)^{\nu_P(f)}|}.$$

2.8 End of Proof

This now becomes very simple. Without loss of generality, we assume that there is only one place, a complex one, and we view $k(X_F) \otimes_{\mathbb{Q}} \mathbb{R}$ as a subspace of $\mathbb{C}((u))$. Then we are led to the central extension $\mathbb{C}((u))_{\mathbb{C}[[u]]}^*$, consisting of elements (f, α) where $f \in \mathbb{C}[[u]]$ and $\alpha \in [\mathbb{C}[[u]] \mid f\mathbb{C}[[u]]]$. Construct a natural metric on $\mathbb{C}((u))$ using the standard metric on \mathbb{C} and $(u^i, u^j) = \delta_{ij}$. Accordingly, by the detailed calculations carried out in §A.2.6, with the trivial metrics on $\mathbb{C}((u))$ and hence over various spaces used, we have

$$[f|g]_{\mathbb{C}[[u]]} = -\log \frac{|f_0(P)|^{\nu_P(g)}}{|g_0(P)|^{\nu_P(f)}}.$$

This then ends our proof.

Remark. Note that the appearances of $-\log |f(P)|$ is not surprising. Indeed, when we numerate spaces $W_{\mathcal{L}}^{\text{ar}}$, we use the spaces $W_{\mathcal{L},s}^{\text{ar}}$ defined by

$$W_{\mathcal{L},s}^{\text{ar}} = \left(\prod_{x \in E_P} \mathfrak{m}_{E_P,x}^{-n_x}((u)) \right) \times \left(\prod_{Q \in X_F} \mathcal{O}_{X_F}(Q)^{\otimes m_Q} |_P((u)) \otimes_{\mathbb{Q}} \mathbb{R} \right) \times \mathbf{e} \left(\int_{X_{\infty}} -\log \|s\| d\mu \right).$$

It is the additional factor $\mathbf{e} \left(\int_{X_{\infty}} -\log \|s\| d\mu \right)$ which makes the whole intersection theory and hence our construction work well. For example, the

canonical numeration num_0 for the space $W_{\overline{\mathcal{L}},s}^{\text{ar}}$ is given by

$$\begin{aligned} & \text{num}_0 \left(\left(\prod_{x \in E_P} \mathfrak{m}_{E_P,x}^{-n_x} \right) \times \left(\prod_{Q \in X_F} \mathcal{O}_{X_F}(Q)^{\otimes m_Q} \Big|_P \widehat{\otimes}_{\mathbb{Q}} \mathbb{R} \right) \times \mathbf{e} \left(\int_{X_\infty} -\log \|s\| d\mu \right) \right) \\ &= \prod_{x \in E_P} q_x^{n_x} \prod_{Q \in X_F} G_\infty(P, \text{div}(s_F)) \cdot \mathbf{e} \left(\int_{X_\infty} -\log \|s\| d\mu \right). \end{aligned}$$

In particular, when $\overline{\mathcal{L}} = \overline{\mathcal{O}}_X(\text{div}_{\text{ar}}(f))$, then $\text{num}_0(W_{\overline{\mathcal{O}}_X(\text{div}_{\text{ar}}(f)),f}^{\text{ar}}) = 0$ by the product formula. Consequently, for different choices of rational sections s and s' of \mathcal{L} , we have canonical isometry

$$\text{Num}(W_{\overline{\mathcal{L}},s}^{\text{ar}}) \cong \text{Num}(W_{\overline{\mathcal{L}},s'}^{\text{ar}}).$$

In particular, it makes sense to introduce the space $W_{\overline{\mathcal{L}}}^{\text{ar}}$. As a direct consequence, for rational function $f \in k(X_F)^*$, we obtain this crucial term $-\sum_{\sigma|\infty} \log |f_\sigma|_\sigma$ and hence the reciprocity symbol at infinite.

References

- [1] E. Arbarello, C. De Concini, and V.G. Kac, The infinite wedge representation and the reciprocity law for algebraic curves, Proc. of Symp in Pure Math, Vol. 49 (1989), Part I, 171-190
- [2] A.A. Beilinson, Residues and adeles, Funct. Anal. Pril., 14 (1980), no. 1, 44-45; English transl. in Func. Anal. Appl., 14, no. 1 (1980), 34-35.
- [3] J.-L. Brylinski and P. Deligne, Central extensions of reductive groups by K2, Publ. Math. Inst. Hautes Etudes Sci. No. 94 (2001), 5-85.
- [4] S. Lang, *Introduction to Arakelov Theory*, Springer (1988)
- [5] M. Kapranov, Semi-infinite symmetric powers, math.QA/0107089
- [6] D. V. Osipov, Central extensions and reciprocity laws on algebraic surfaces, Mat. Sb. 196 (2005), no. 10, 111-136; translation in Sb. Math. 196 (2005), no. 9-10, 1503-1527, arXiv:math/0501155
- [7] D. V. Osipov, Unramified two-dimensional Langlands correspondence, Izv. Ross. Akad. Nauk Ser. Mat. 77 (2013), no. 4, 73-102; translation in Izv. Math. 77 (2013), no. 4, 714-741, arXiv:1210.3780
- [8] D. V. Osipov and X. W. Zhu, A categorical proof of the Parshin reciprocity laws on algebraic surfaces, Algebra Number Theory 5 (2011), no. 3, 289-337

- [9] A.N. Parshin, On the arithmetic of two-dimensional schemes. I. Distributions and residues. *Izv. Akad. Nauk SSSR Ser. Mat.* 40 (1976), no. 4, 736-773
- [10] A.N. Parshin, Chern classes, adeles and L -functions. *J. Reine Angew. Math.* 341 (1983), 174-192.
- [11] K. Sugahara and L. Weng, Arithmetic cohomology groups, arXiv:1507.06074 (2013)
- [12] L. Weng, Geometry of numbers, arXiv:1102.1302
- [13] J. Tate, Residues of differential forms on curves, *Ann. Sci. Eco. Norm, Sup.*, 4 serie, t. 1, 1968, 149-159

K. Sugahara & L. Weng
 Graduate School of Mathematics,
 Kyushu University,
 Fukuoka, 819-0395,
 Japan
 E-mails: k-sugahara@math.kyushu-u.ac.jp,
 weng@math.kyushu-u.ac.jp